Learning Sequences in the Acquisition of Mathematical Knowledge: Using Cognitive Developmental Theory to Inform Curriculum Design for Pre-K–6 Mathematics Education

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ABSTRACT—Using Central Conceptual Structure theory as an heuristic, learning sequences in the acquisition of number knowledge are described in three forms that bridge theory and practice: as a four-stage theoretical progression, as items on a developmental test created to test the theoretical progression, and as learning objectives in a curriculum designed to teach math concepts and skills implied in the theoretical progression. Other aspects of the theory that were used to create teaching methods and materials for the Number Worlds curriculum are also described, as are some of the outcomes of this theory-based program.

What does the label "research-based" mean when it is used to describe mathematics curricula and is often prominently displayed on the title pages of these curricula? An answer to this question is typically not provided in the curriculum materials and one is left to wonder whether the claim that the program is "research-based" is entirely vacuous and made solely to sell the program, or whether it assumes that, if development of the program was funded by an educational foundation, it must, in some fashion, be research based, or whether the program is genuinely grounded in an (often unidentified) theory and/or body of research. To help buyers of these programs make informed decisions on the quality of the program they are considering using in their schools and to help researchers identify the value of various theoretical paradigms for the development of educational materials, there is an urgent need for the links between theory and practice to be made more visible.

In the present article, I describe the way one particular theory, central conceptual structure theory (see Case, 1996; Griffin, 2005a; Griffin & Case, 1997, for a review of this theory) was used to construct one particular PreK-6 mathematics curriculum called Number Worlds (Griffin, 2007a). In so doing, I describe learning sequences in the acquisition of mathematical knowledge at three levels of generality. Starting with the theory that informed curriculum development, I describe these sequences, first, in terms of the four-stage theoretical progression that was constructed to explain the development of mathematical knowledge across the age range of 4–10 years. Moving to a more concrete level, I describe these sequences, next, in terms of items on a developmental test—the Number Knowledge test—that were constructed to provide operational definitions for the age-level theoretical postulates and a means to assess their validity. Finally, at the most concrete level, I describe these sequences in terms of specific learning trajectories for school-based math concepts and skills, as suggested by research within this theoretical paradigm.

At each level of generality, the learning sequences contain useful information that was systematically relied upon to

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identify mathematical learning goals for the Number Worlds curriculum, the way these goals should be sequenced within the program, and the way they should be allocated to grade-level categories. To address another curriculum question that is almost as important as the one just mentioned, namely "How should this knowledge be taught?", it was necessary to move beyond postulates of the theory that explain *what* changes in cognitive development and to examine postulates that explain *why* developmental change occurs and *how* developmental change occurs. These aspects of the theory were used to create and/or select teaching methods and materials to foster developmental change (i.e., the acquisition of mathematical knowledge) and they are described in subsequent sections of this article. Finally, the relative merits of the particular theory-based curriculum that is described in this article vis-à-vis other, more conventional curricula, are discussed.

THE DEVELOPMENT OF MATHEMATICAL KNOWLEDGE: A THEORETICAL FRAMEWORK

The theory that was used for this endeavor, central conceptual structure theory (Case & Griffin, 1990; Case & Okamoto, 1996), is one of a family of neo-Piagetian theories (e.g., Demetriou, Efklides, & Platsidou, 1993; Fischer, 1980; Halford, 1982) that sought to retain the strengths of Piaget's theory and to remedy its weaknesses. As well as retaining many of Piaget's theoretical postulates, which are described further in later sections, central conceptual structure theory added three new postulates, which made Piaget's theory much more applicable to education.

The first was to suggest that, although the form of children's thought remains constant across content domains (as Piaget himself proposed), the content of children's thought differs across content domains and can be specified by determining the manner in which concepts that are central to disciplinary understanding increase in complexity as children mature. Several developmental progressions were identified for content domains relevant to schooling (e.g., mathematical understanding, narrative understanding, spatial-artistic understanding, and social understanding) and for the age range of 4–10 years, and it is the first of these that is of interest to the present article (see Case, 1992 for a review of these progressions). The second new postulate described substage transitions across this age range more precisely than Piaget had done and suggested that these occur, on average, about every 2 years. The third new postulate proposed a mechanism—growth in working memory—to explain the rate of conceptual change and the fact that children's thought tended to assume a common form (i.e., a certain level of complexity) across content domains. As it will be seen, each of these postulates was used to guide development of the Number Worlds program.

The developmental progression that was identified for mathematical understanding is presented in Figure 1. As the figure suggests, at the age of 4 (i.e., at some point between the ages of 3 and 5) children have two conceptual structures available to them, which are not yet merged. The first enables them to represent all possible variables in a global or polar fashion, so they can make mappings of the sort: "Big things are worth a lot; little things are worth a little." The second, which is used independently of the first, enables them to count small sets of objects. At the age 6 years (i.e., typically between the ages of 5 and 7), with the hierarchic integration of these two structures, which marks transition into the next major developmental stage, children can represent variables in a continuous fashion (i.e., as having two poles and a number of points in between). Moreover, they realize that these points can be treated as lying along a mental number line, such that values that have a higher numeric value also have a higher real value associated with them. At the age of 8 (i.e., typically between the ages of 7 and 9), children can think in terms of two independent quantitative variables (e.g., tens and ones in the base ten system; hours and minutes on a clock; dollars and cents) but cannot yet make successful comparisons between variations along each. Finally, at the age of 10 (i.e., typically between the ages of 9 and 11), children can make these sorts of comparisons, by thinking in terms of the "trade-offs" between two quantitative variables.

Although the figure does not illustrate the conceptual structures children construct in the next major developmental stage, the theory suggests that, at the age of 12 (i.e., typically between the ages of 11 and 13), children become capable of representing the relationships between two fully developed conceptual structures from the previous stage and achieving, for example, a rich understanding of fractions, decimals, percents, and functions (see Kalchman, 2001; Moss & Case, 1999 for descriptions of this level of mathematical development and support for this claim).

Before proceeding it is important to note that the ages suggested in the theoretical progression are based on normative data from children who have access to the learning opportunities that are typically available in developed societies (i.e., children from middle income families). As it will be discussed later, the theory allows for considerable variability in the rate at which individual children progress through this sequence and would predict, for example, that children with enriched learning opportunities would achieve the developmental milestones at an earlier age than children from impoverished communities and that children with special talents in one area of mathematics (e.g., spatial reasoning) might demonstrate accelerated growth on tasks that require this form of reasoning. In short, although it is useful to know the age ranges at which the "average" child achieves developmental competencies, age itself, in the absence of

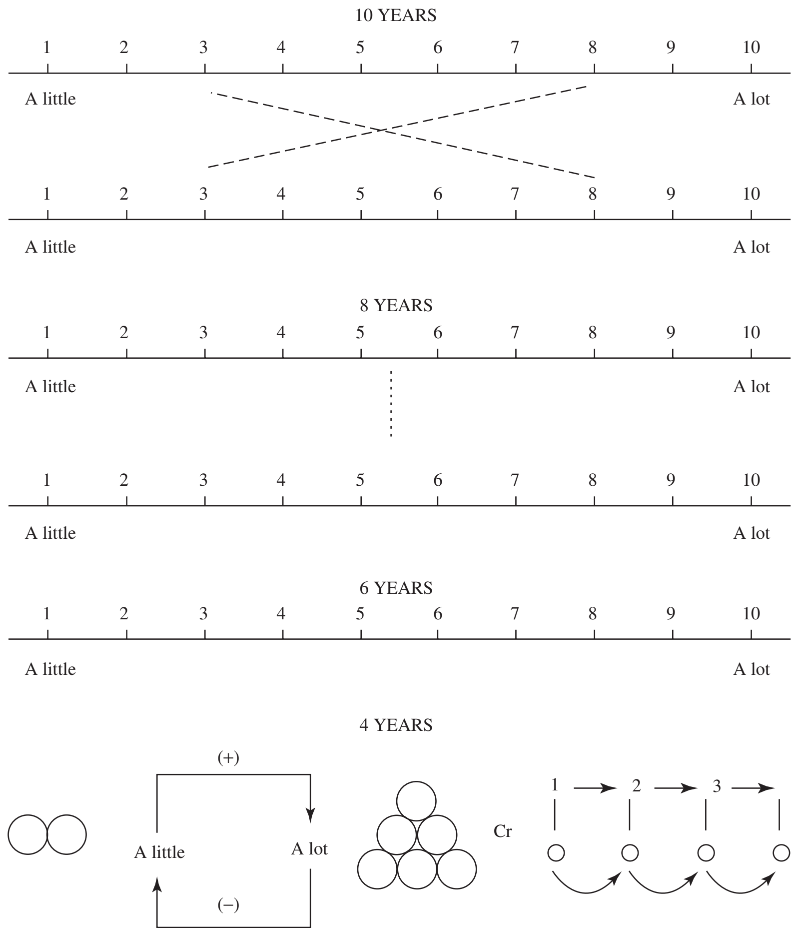

Fig. 1. Development of the central numerical structure.

other considerations such as quality of experience, cannot be used as an index or a predictor of development.

This progression can be further illustrated by describing children's performance on one of the many tasks drawn from the developmental research literature that were used to construct it. On Siegler and Shrager's (1984) balance beam task, 4-year-old children are able to predict which side of the beam will go down if stacks of weights placed on each side (equidistant from the fulcrum) differ sufficiently in number to be visible to the naked eye. Although children can count at this age, they do not use their counting skills to solve these problems but rather, rely on perceptual cues alone and predict that the beam will balance if the number of weights on each side differs slightly. This performance persists even with proof that their prediction was wrong.

By the age of 6, children spontaneously use their counting skills to count the number of weights, even when the differences are visible to the naked eye, and rely on the results of counting to make their prediction. At this age, however, if the number of weights on each side is identical but the stacks of weights are placed at different distances from the fulcrum, children pay no attention to the second dimension (distance) and predict that the beam will balance. This performance persists even with proof that their prediction was wrong and even when they are visibly disturbed that the problem-solving strategy they are using is not working. By 8 years, children spontaneously attend to the second dimension and count both the number of weights in each stack and the number of pegs between the fulcrum and the weights (which provides an index of distance) to make their prediction. At this age, however, if the number of weights in each stack and the distance from the fulcrum both vary, they are unable to compensate for differences in one dimension when considering the effects of the second dimension. By 10 years, children are able to consider the "trade-off" between these two dimensions and solve this sort of problem.

This example illustrates an important feature of central conceptual structure theory. The developmental progressions it proposes describe children's natural or spontaneous development and the age-level capabilities children are able to demonstrate in situations where adult scaffolding and support is not provided. For purposes of instructional planning, this feature of the theory makes sense to the present author. It enables one to select instructional objectives that are "developmentally appropriate" and that are known, at least in their level of cognitive complexity, to be within the intellectual capabilities of the majority of children. The theory also suggests that, with adult guidance and support, which can reduce the working memory demands of any task and make it easier to process, children may be capable of performing at one substage higher than they are capable of achieving on their own (see McKeough, 1992, for support for this claim). Although it is useful to understand the limits of what one can teach at any age, using upper limits to establish educational objectives for the majority of children, as some curriculum planners do, does not seem a wise decision to this author.

Although the theoretical progression that has just been described is a bare-bones description of growth in mathematical understanding that does not specify the learning sequences children go through to achieve each developmental milestone, its simplicity serves a valuable purpose by highlighting major age-related changes in children's mathematical development. The present author has used this progression again and again to make global curricular decisions, such as the most appropriate age or grade level to teach any particular math concept or skill. For example, because a good understanding of double-digit arithmetic, of multiplication, of time telling (hours and minutes on an analog clock), and of monetary systems (dollars and cents) requires an ability to coordinate two quantitative dimensions, it makes sense to postpone the teaching of these concepts and skills, with the expectation of mastery, until the second grade (or later, depending on the complexity of the particular concepts to be taught), when the majority of children will be 7–8 years old. The wisdom of this recommendation is readily apparent if one visits any first-grade classroom where time telling to the hour and minute is being taught (to comply with state standards) and witnesses the struggle and the frustration that is experienced by teacher and students alike, and the ultimate failure of most students to achieve success. When the same concept is taught a year later, children master it easily, as the theory would suggest (Griffin, 2007b).

The ideal age or grade level to teach math concepts that do not rely heavily on number knowledge (e.g., geometry concepts) is not always easy to predict from the mathematical progression just described and for these purposes, combining the insights that are available in the developmental progressions that have been identified for spatial-artistic understanding and/or for narrative understanding with those that are available for mathematical understanding can help one make more informed and precise estimates of the best time to teach these concepts. Describing the use of the theory for these purposes is beyond the scope of the present article.

THE NUMBER KNOWLEDGE TEST: AN INTERMEDIATE MODEL OF THE DEVELOPMENTAL PROGRESSION

To determine the extent to which the theoretical progression that has just been described can predict the development of mathematical knowledge that is more directly related to schooling, a set of number knowledge problems that tap the understandings postulated at each age level in the theory was constructed. The first four levels of this instrument, called the Number Knowledge test (Case & Griffin, 1990; Case & Okamoto, 1996; Griffin, Case, & Sandieson, 1992) are presented in Table 1. As well as providing a tool to test the theory, this test provides an application of the theory to education and a model of the development of mathematical understanding that is intermediate between the theoretical progression just described and the more detailed learning sequences that are described in a later section. Because the test was designed to assess the deep conceptual knowledge of number that is reflected in the theory and not more fragile and superficial understandings that can be achieved by direct instruction, it differs from conventional math tests in several respects. For example: (a) test questions were designed to be novel tasks to minimize the extent to which performance would be influenced by previous instruction and (b) the test was designed to be an oral test to tap conceptual understanding and working memory capacity and to minimize the procedural supports that can be obtained through the use of paper and pencil. These features and other novel features of the test can be illustrated by describing the reasoning that guided construction of items at each level and, for certain items, by describing the manner in which children responded when the test was used in several studies.

The items included at level 0 (so named because this level is hypothesized to precede and to lay a foundation for the major stage transition that occurs at the next level) are straightforward applications of the theoretical model. They test ability to count small sets of objects (items 1, 4, and 5) and ability to use perceptual cues to make global quantity judgments (items 2 and 3). Unlike items included at all higher levels and consistent with the thinking that is hypothesized for this level, concrete manipulatives are provided for each item.

At level 1 (so named because this level represents the first substage in the dimensional stage of development), children are believed to have constructed a mental number line and to be able to make quantitative determinations on the basis of number alone, without concrete objects. Items included at this level assess children's understanding of the single-digit number sequence (items 2, 3, 6, and 9), their ability to make relative size judgments (items 4 and 5), and their ability to use their knowledge of counting to add and subtract small sets (items 1, 7, and 8). Note that items *within* this level and all higher levels are not sequenced by level of difficulty but rather,

Table 1
Number Knowledge Test

Level 0 (4-year-old level): Go to level 1 if 3 or more correct

1. Can you count these chips and tell me how many there are? (Place 3 counting chips in front of child in a row)
2a. (Show stacks of chips, 5 vs. 2, same color). Which pile has more?
2b. (Show stacks of chips, 3 vs. 7, same color). Which pile has more?
3a. This time I'm going to ask you which pile has less.
(Show stacks of chips, 2 vs. 6, same color). Which pile has less?
3b. (Show stacks of chips, 8 vs. 3, same color). Which pile has less?
4. I'm going to show you some counting chips (Show a line of 3 red and 4 yellow chips in a row, as follows: R Y R Y R Y Y). Count just the yellow chips and tell me how many there are.
5. Pick up all chips from the previous question. Then say: Here are some more counting chips (Show mixed array [not in a row] of 7 yellow and 8 red chips). Count just the red chips and tell me how many there are.

Level 1 (6-year-old level): Go to level 2 if 5 or more correct

1. If you had 4 chocolates and someone gave you 3 more, how many chocolates would you have altogether?
2. What number comes right after 7?
3. What number comes two numbers after 7?
4a. Which is bigger: 5 or 4?
4b. Which is bigger: 7 or 9?
5a. This time, I'm going to ask you about smaller numbers. Which is smaller: 8 or 6?
5b. Which is smaller: 5 or 7?
6a. Which number is closer to 5: 6 or 2? (Show visual array after asking the question)
6b. Which number is closer to 7: 4 or 9? (Show visual array after asking the question)
7. How much is 2 + 4? (OK to use fingers for counting)
8. How much is 8 take away 6? (OK to use fingers for counting)
9a. (Show visual array −8 5 2 6—and ask child to point to and name each numeral). When you are counting, which of these numbers do you say first?
9b. When you are counting, which of these numbers do you say last?

Level 2 (8-year-old level): Go to level 3 if 5 or more correct

1. What number comes 5 numbers after 49?
2. What number comes 4 numbers before 60?
3a. Which is bigger: 69 or 71?
3b. Which is bigger: 32 or 28?
4a. This time I'm going to ask you about smaller numbers. Which is smaller: 27 or 32?
4b. Which is smaller: 51 or 39?
5a. Which number is closer to 21: 25 or 18? (Show visual array after asking the question)
5b. Which number is closer to 28: 31 or 24? (Show visual array after asking the question)
6. How many numbers are there in between 2 and 6? (Accept either 3 or 4)
7. How many numbers are there in between 7 and 9? (Accept either 1 or 2)
8. (Show card 12 54) How much is 12 + 54?
9. (Show card 47 21) How much is 47 take away 21?

Level 3 (10-year-old level): Go to level 4 if 4 of more correct

1. What number comes 10 numbers after 99?
2. What number comes 9 numbers after 999?
3a. Which difference is bigger: the difference between 9 and 6 or the difference between 8 and 3?
3b. Which difference is bigger: the difference between 6 and 2 or the difference between 8 and 5?
4a. Which difference is smaller: the difference between 99 and 92 or the difference between 25 and 11?
4b. Which difference is smaller: the difference between 48 and 36 or the difference between 84 and 73?
5. (Show card, "13, 39") How much is 13 + 39?
6. (Show card, "36, 18") How much is 36 − 18?
7. How much is 301 take away 7?

are grouped and ordered in a logical sequence to facilitate ease of comprehension of the test questions. Note also that two items were constructed for all forced choice questions (that could be answered by guessing) with the requirement that children must pass both items to receive a pass score. One final item at this level is also worth mentioning. Consistent with the tasks Piaget constructed, item 9 (a and b) tests children's understanding of the counting sequence in the presence of conflicting cues (i.e., the written numbers on the visual display) and thus, serves as a measure of the strength of their understanding of this sequence (see Wadsworth, 1996, for descriptions of several Piagetian tasks, including his conservation tasks which are prime examples of tasks with conflicting cues).

Items at level 2 (so named because this level represents the second substage in the dimensional stage of development) parallel the format of items at level 1 but at a higher level of complexity. They all tap double-digit understandings and/or ability to use two quantitative dimensions. The relative size questions (items 3 and 4) require that children attend to the value of the tens digits to determine which is bigger or smaller and, because the ones digits are larger in magnitude than the tens digits, they can present conflicting cues to children whose base ten understandings are not well developed. These children, as well as most children below the age of 7, respond to these questions by attending only to the value of the numbers in the ones place. Items 6 and 7 were included at this level because it was believed that computing the number of numbers between other numbers requires the use of two mental number lines: one to represent a number line on which the upper and lower markers can be noted and another to count the numbers in between these markers. If a copy of a number line were physically available to children, the theory would predict that these two items could be passed at an earlier level, using a single mental number line conceptual structure.

The arithmetic items (items 8 and 9) require that children use their knowledge of tens and ones to mentally calculate the sum and difference of two 2-digit numbers and this is easily achieved if children compute the tens value first (e.g., by skip counting) and then add or subtract the ones value. Younger children and children who have not yet constructed base ten understandings frequently attempt to simplify the problem by creating a mental image of the numbers aligned vertically (i.e., in a typical workbook fashion), which permits them to deal with each column of numbers separately and, in essence, reduces a double-digit number problem to two single-digit problems. This sometimes works but the heavy procedural demands of this strategy often results in errors. As these examples illustrate, the level of complexity of the test questions (and the demands they place on working memory) is highly related to the age at which particular competencies (e.g., base ten understandings) can be demonstrated and it is easy to get a false reading of these competencies if the tasks can be solved with more primitive understandings. This is an important fact to keep in mind when constructing learning sequences that will form the basis for instructional objectives, and this effort can be facilitated by careful attention to the cognitive demands of the tasks used to teach and/or assess any particular concept or skill.

Items at level 3 (so named because this level represents the third and final substage in the dimensional stage of development) parallel the format of items at previous levels but the complexity of these items has been increased to assess ability to handle number sequence problems involving three- and four-digit numbers (items 1 and 2) and arithmetic problems (items 5, 6, and 7) that require re-grouping (i.e., that require that trade-offs between two quantitative dimensions be considered). The relative size questions (items 3 and 4) use single-digit numbers but the demands of these tasks (i.e., that differences between two sets of numbers be computed and held in memory so that these differences themselves can be compared) require that two quantitative dimensions be considered in an integrated fashion.

At each level of the test, questions to assess strategy use (e.g., How did you figure that out?) were also included for selected items (e.g., items 1 and 3 at level 1). These questions provide another layer of information that can be used to fine-tune assessments of children's developmental level and/or problem-solving capabilities. For example, although the theory predicts that children can solve the 4 + 3 question posed at level 1 around the age of 6 (i.e., between the ages of 5 and 7 years), research with this test has shown that this question can be correctly solved at the lower limits of this age range by using fingers and the count-all strategy. Use of more sophisticated strategies (e.g., the count-on strategy and the retrieval strategy) does not typically occur until a year or so later (Griffin, 2003). These findings make it possible to create more finely graded learning sequences than are suggested by the intermediate model alone and they were used to construct the learning sequences (described in a later section) that formed the backbone of the Number Worlds curriculum.

Before closing this discussion of intermediate models, it should be noted that the theoretical progression has also been used to construct developmental tests of time-telling knowledge and money-handling knowledge that are described in several publications (e.g., Griffin, 2002; Griffin et al., 1992). Like the Number Knowledge test, these tests provide intermediate models of the development of mathematical understanding, and they were used by the present author to develop learning sequences for these concepts and to construct the Number Worlds curriculum.

TESTING THE INTERMEDIATE MODEL: ASSESSING ITS DEVELOPMENTAL VALIDITY AND EDUCATIONAL UTILITY

A number of studies were conducted to assess the reliability and validity of the Number Knowledge test. In the first study, designed to answer the question, “Do the items included at four levels of the test provide an accurate and reliable developmental scale?” the test was administered to 4-, 6-, 8-, and 10-year-old children in a cross-sectional research design. The results indicated that the majority of children passed a majority of items at the predicted level. A Guttman scale analysis also showed the presence of a strong developmental progression. With the criterion for passing a level set at 60%, the coefficients of reproducibility and scalability were both 1.00. Note that similar findings were found for the time-telling and money-handling tests (Griffin et al., 1992). Note that Knight and Fischer (1992) have also used Guttman scaling techniques to build learning sequences in the area of reading.

The next series of studies were designed to assess the extent to which items included at the 6-year-old level of the test assessed knowledge that was (a) central to performance on a range of quantitative tasks and (b) central to further development in this domain, as predicted by central conceptual structure theory. The Number Worlds program for kindergarten was developed to test these hypotheses and, in a series of training studies that carefully avoided providing instruction in time-telling and money-handling skills, kindergartener’s performance on the Number Knowledge test was found to be highly related to performance on a battery of quantitative tasks that included the time-telling test, the money-handling test, Siegler and Shrager’s balance beam test, Damon’s distributive justice test, and Marini’s birthday party test (Griffin, Case, & Siegler, 1994). These findings provided strong support for the first part of the theoretical prediction and, in so doing, suggested that the 4- and 6-year-old levels of the test had good content validity (i.e., the items assessed the central conceptual understandings specified in the theory).

The second part of the theoretical prediction—that knowledge assessed on the Number Knowledge test was foundational to further development—was assessed in a follow-up study (Griffin & Case, 1996; Griffin et al., 1994) in which graduates of the Number Worlds kindergarten program and children in a matched control group, all of whom had spent grade 1 in a variety of more traditional classrooms, were given a series of developmental and traditional math tests at the end of grade 1. In contrast to the control group, low-income children who had received the Number Worlds program in kindergarten started grade 1 with mastery of the knowledge assessed at level 1 of the Number Knowledge test. At the end of the year, these children performed significantly better than the control group on a word problems test, on tests of written and oral arithmetic, and on teacher ratings of number sense and several other aspects of mathematical reasoning. These findings provided support for the second part of the theoretical prediction. They also suggest that the Number Knowledge test, when administered at the end of kindergarten, has considerable educational utility in predicting success in school math in grade 1.

Results of a longitudinal study, depicted in Figure 2, provide further support for the validity of the Number Knowledge test. In this study (Griffin, 2005b; Griffin & Case, 1997), three groups of children (a low-income group who received the Number Worlds program in kindergarten and in grades 1 and 2, a matched control group who received a variety of other math programs, and a normative group drawn from a normally distributed population) were followed for a period of 4 years. The Number Knowledge test (as well as a variety of other tests) was administered to all children at the beginning of kindergarten and at the end of kindergarten and grades 1, 2, and 3. As the figure illustrates, the performance of the normative group was consistent with theoretical expectations at the beginning of kindergarten and at the end of kindergarten

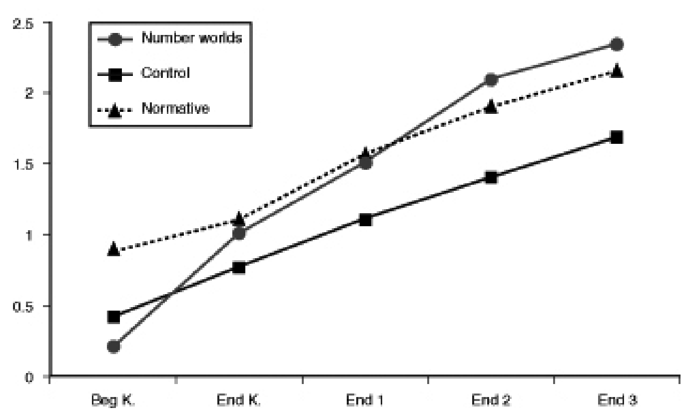

Fig. 2. Mean developmental level scores on the Number Knowledge test at five age periods.

and grades 1 and 2 (e.g., children had a mean age of 6 at the end of kindergarten and achieved a mean developmental level score of 1.0 as predicted by the theory) and was somewhat lower than theoretical expectations at the end of grade 3.

By contrast, the performance of the control and the treatment groups was at least 1 year below theoretical expectations for normative populations at the beginning of kindergarten. This finding can also be explained by the theory by suggesting that opportunities for mathematical exploration and social interaction—factors postulated to facilitate conceptual change—were less available to these (low-income) children during their preschool years. The performance of the control group was consistently lower than the performance of the normative group at each subsequent grade level (with significance of these differences found to be at the 0.001 level using analysis of variance (ANOVA) analyses and at the 0.05 level or better using the Scheffe procedure), suggesting that performance on the Number Knowledge test at the beginning of kindergarten predicts performance on this test up to 4 years later. This finding provides additional evidence that the knowledge assessed with the Number Knowledge test is foundational to further development in this domain. Finally, the performance of the Number Worlds group suggests that the pattern demonstrated by the control group can be altered with a theory-based instructional program, which not only enabled these low-income children to perform as well as (or better than) the normative group at the end of the first 3 years of schooling (when their exposure to the Number Worlds program ended) but that also enabled them to maintain the gains they achieved 1 year later, at the end of grade 3.

LEARNING SEQUENCES IN THE ACQUISITION OF MATH CONCEPTS AND SKILLS

In order to develop the Number Worlds PreK-6 curriculum and to ensure that it was consistent with children's developing capabilities (i.e., the manner in which most children construct mathematical knowledge), the theoretical progressions described in previous sections were used to create a large number of learning sequences for a variety of math concepts and skills (e.g., number sense, algebra, geometry, and measurement concepts) and for a wide age range (i.e., 4–12 years). These progressions formed the backbone of the program and were used to create sequences of learning objectives and to specify the grade levels at which particular objectives could most easily be taught and mastered. To illustrate this process, the learning sequences that were identified for a few number and quantity concepts and for a limited age range are presented in Table 2.

Table 2
Learning Sequences in the Acquisition of Early Number and Quantity Concepts

The ***Big Idea*** children acquire in this area is the realization that the counting numbers can be used with unfailing precision, not only to figure out how many objects are in a set, but also to figure out how many objects there will be if you add *X* more to the set or if you take *X* away from the set. All you need to do to figure this out is to start with the number you already have in the set and count up by the number you are adding or count back by the number you are subtracting. The relationship between numbers and quantity is central to this Big Idea.

Quantity concepts (*level 0*)

1. Recognizes perceptually salient differences in sets and can use words to describe these differences, such as lots–little, big–small, high–low, long–short, tall–short, heavy–light, more–less, farther along–less far along.
2. Understands that a set gets bigger when objects are added to it and smaller when objects are taken away and can use words to describe these transactions, such as more–less or more–fewer.

Verbal counting (*level 0*)

1. Counting up: Can say the counting string from one to five (and later to ten and still later to twenty).
2. Alternate counting: Can pick up counting where someone else left off and say the next number (or the next few numbers) up in the sequence.
3. Given any number in the one to ten sequence, can say the next number up.
4. Counting down: Can perform each of the previous skills for the five to one (and later the ten to one) count down sequence.

Counting objects and position markers (*level 0–1*)

1. Knows that you touch each object once and only once while counting.
2. Knows that the last count number you said tells you how many are in the set.
3. Can count out collections of a specified size (e.g., get 5 plates for our picnic).
4. Can count a partial set (e.g., just the red objects), keep track of which objects have been counted (without separating the sets) and tell you how many there are.

Table 2
(*Continued*)

5. Knows that counting can be used to enumerate a range of other entities besides objects, such as the fingers on your hands; the dots on dice and playing cards; the steps you take along a path; the houses you pass in a neighborhood; the lines or spaces on a thermometer or bar graph; the points or spaces on a dial.
6. Knows that numbers are represented in different ways: as groups of objects, as finger displays, as patterns on dice or dominoes, as steps along a path, as points (or spaces) on a thermometer or a dial. Even though the display differs dramatically, five is five in each of these contexts.
7. Understands that numbers indicate position in a sequence (the ordinal value of the number) as well as how many there are (the cardinal value) and that these two values are related. For example, if you take five steps on a numbered path, you are on the fifth step from the start and you have taken five steps in all.
8. Can add two small sets by starting at one and counting all items in the combined sets.
9. Can use the count-on strategy (starting the count with the quantity in one of the sets and counting on the items in the second set) to add two sets if the number being added is small (e.g., 1, 2, or 3) and/or if both sets are visible.

When the above competencies have been developed and are well consolidated, children become capable of constructing the Big Idea described at the beginning of this section. When this understanding is in place, children can develop and master more advanced number skills and concepts, including those listed below.

Number sense (*level 1*)

1. Knows that counting yields a more precise estimate of set size than perceptual cues alone.
2. Knows that numbers that come later in the counting sequence (or are higher up the sequence) are bigger than numbers that come earlier.
3. Knows that counting can be used to compare set size and the set with the bigger number is the larger set.
4. Knows that numbers that have been assigned to sets (to indicate how many) can be used (without the need for counting) to compare sets along any variable, such as length, height, weight, and the set with the larger number has the greater value.

Operation sense (*level 1*)

1. Can use counting, without the presence of physical objects, to predict how many there will be if X (a small number) is added to or taken away from a set.
2. Can use counting, without the presence of physical objects, to predict how many more you will need to reach a target number that is 10 or smaller.
3. Can use counting to tell you how many more one set has than another (by counting up from the smaller set).
4. Can mentally add two (and later three) small numbers (usually by counting up from the larger number).
5. Can mentally subtract a small number from a larger number that is 10 or smaller (either by counting back from the larger number or by using knowledge of the counting sequence to identify the number that you could count up from to reach the target number).
6. Knows that each number from 2 to 10 is composed of other numbers (e.g., 6 is $5 + 1$; $4 + 2$; $3 + 3$).

Measurement sense (*level 1*)

1. Can use counting and nonstandard measures (e.g., hands, footsteps, learning links), to measure objects along several dimensions, such as length, width, height.
2. Can compare and order a set of measurements by size. This task is made much easier if physical representations of the measurements obtained (e.g., markers on a wall to indicate the heights of various students) are made available.

Written representations for numbers (*level 1–2*)

1. Knows the symbols for each number in the 1–10 (and later the 1–20) sequence. (*Note:* Symbolic notation is often introduced in kindergarten. Research suggests it is much easier to master if it is introduced *after* children have acquired a solid understanding of the meaning of the counting names).
2. Knows the symbols for addition, subtraction, and equality. (See above comment. These abstract symbols are much easier to master if they are linked to names and operations that children thoroughly understand).
3. Can create and interpret a variety of representations of numbers and quantities, such as tally marks, pictographs, and bar graphs.
4. Can use these representations for a variety of purposes: to record data, such as how many children guessed correctly and incorrectly; to display data; to draw inferences from data.

As the table indicates, networks of understandings that are believed to contribute to the development of particular conceptual structures (e.g., quantity schemas and counting schemas) are listed separately. Separate sequences are also proposed for the development of concepts that are interrelated but that are treated separately in many school curricula (e.g., number sense concepts, number operation concepts, and measurement concepts) to ensure that they will make sense to teachers and be useful for their own instructional planning. For example, because different students and even the same student may be at different places along these different sequences, the provision of separate sequences for specific concepts and skills enables the teacher to fine-tune instruction to meet children's developmental needs. Consistent with the theory, knowledge that is typically acquired between the ages of 3 and 5 years has been labeled "level 0"; knowledge that is typically acquired between the ages of 5 and 7 years has been labeled "level 1"; etc. Within each level and for each category, understandings are listed and numbered in the order in which they are typically acquired. Finally, markers to indicate major stage transitions (i.e., the consolidation and hierarchic integration of previous knowledge) are also included in the description of these learning sequences to alert teachers to the "big ideas" they should be teaching and the major learning goals of the program.

THEORY-BASED TEACHING METHODS AND MATERIALS

As the discussion to date indicates, the Number Worlds program was built on carefully constructed learning sequences that reflect children's natural development. In keeping with the theory that informed program development, every effort was made in designing the curriculum to ensure that children can progress through these sequences at a rate that is comfortable for them and that enables them to consolidate one level of understanding before moving on to the next. Because children in any classroom differ in level of understanding at any one time, putting this principle into practice requires (a) that teachers have tools to assess children's progress on a daily or weekly basis, (b) that the learning activities that are provided provide multiple levels of learning and can be used to differentiate instruction to meet the learning needs of individual children, and (c) that classroom structures are in place to enable different groups of children to pursue different learning goals, ones that are consistent with their current levels of development. Two of the many ways the Number Worlds program attempts to meets these requirements are (a) by using a game format for many learning activities that have several variations and that permit most children in a class to engage in the same family of activities, at different levels of difficulty and (b) by including small group learning structures in each daily lesson.

The factors that are believed to contribute to conceptual change in Piaget's theory and in central conceptual structure theory as well have also been built into the curriculum as fully as possible to facilitate progress through the learning sequences. These are exploration, social interaction, equilibration, and maturation. (Note that virtually all developmental theorists believe these factors play a central role in development, with some theorists [e.g., Bruner, 1964; Cole, 1991; Vygotsky, 1962] giving greater weight to social interaction as an explanatory construct than other theorists. The neo-Piagetian theorists mentioned in an earlier section [e.g., Case, 1992; Demetriou et al., 1993; Fischer, 1980; Halford, 1982] have tended to give equal weight to all four factors.)

The ways these factors are instantiated in the curriculum are illustrated in the following examples. To provide opportunities for exploration, most of the games and activities are "inquiry-based" activities that pose mathematical problems for children to solve and that provide learning tools (e.g., concrete manipulatives and spatial representations of the number system) to support the problem-solving process. To provide opportunities for social interaction, the small group learning format has been built into each lesson and math talk is not only required, it is supported by a number of props (e.g., teacher questions that are scripted in the program manual; question cards for students to use during the activity to prompt reflection, social interaction, and communication).

To facilitate equilibration, teachers are encouraged to ensure that each child is presented with mathematical problems that are one step beyond their current level of functioning and for which the child has the requisite foundational knowledge. Good assessment tools and a carefully graded sequence of activities help teachers implement this principle. Teachers are also encouraged to create a learning environment where mistakes can be discussed without embarrassment and used as sites for learning. The fourth factor, maturation, is also built into the program by its sensitivity to working memory constraints (which is operationalized by careful attention to the age and grade level at which particular concepts are taught) and by features of the program that enable children to strengthen their current working memory capacity (e.g., plenty of opportunity for practice to develop well-consolidated understandings).

The design of curriculum materials was also heavily influenced by the theory. For example, because the theory postulated that the central conceptual structure for number is represented as a mental number line, a variety of number line representations were created for several games and activities at each level of the program to encourage the construction of this mental structure. These assumed a variety of forms (e.g., horizontal number lines to 10 or 100 displayed as numbered steps along a path or road, vertical number lines displayed as positions on thermometers and elevators, number lines to 100 that were stacked in groups of ten and used to number floors and rooms in hotels or apartment buildings, number lines that curved like the snakes and ladders game and thus presented conflicting cues about numerical magnitude) to be consistent with the complexity of the conceptual structure that was being taught and to ensure that children constructed a mental structure that had a wide range of application.

Note that strong support for the "mental number line" theoretical postulate is provided in recent cognitive neuroscience research which suggests (a) that magnitudes are coded by groups of neurons that are specifically tuned to detect certain numerosities and that are spatially distributed in the brain in a line-like fashion (Dehaene, 1997; Nieder, Freedman, & Miller, 2002), (b) that the representation of symbolic number has strong spatial associations in the human brain (e.g., small numbers are associated with the left side of space and large numbers are associated with the right side of space), and (c) that this association is culturally influenced (e.g., the number line representation of Iranian subjects trained to use east Arabic digits had an opposite spatial direction to that of French subjects trained to use Arabic digits and to write from left to right; Dehaene, Bossini, & Giraux, 1993). More direct support for the use of number lines to teach mathematics in school is provided in a recent training study by Booth and Siegler (2008), which demonstrated that students who used number lines during instruction made greater gains on standard math tests than students who were not given this opportunity.

SUMMARY AND CONCLUSIONS

The findings presented in this article suggest that a mathematics curriculum (Number Worlds) that was originally created to test a theoretical model has considerable educational utility in enabling children who are not meeting developmental expectations to acquire knowledge that is foundational for success in school mathematics. The findings also suggest that a test (the Number Knowledge test) that was created to provide an intermediate model of the theory and to identify learning sequences that are relevant to schooling has considerable predictive power and thus can provide a useful educational tool. Finally, the findings suggest that the theory that was described in this article (central conceptual structure theory) has considerable generative power and can be used productively to develop educational applications (i.e., a tool to assess mathematical knowledge and a program to teach mathematical knowledge) that are central to the mission of schooling.

In this article, learning sequences in the acquisition of mathematical knowledge were described at three levels of generality. At the most abstract level, they were described as a four-stage theoretical progression. At a more concrete level, they were described in terms of test items that provide operational definitions for the age-level theoretical postulates. Finally, at the most concrete level, they were described in terms of sequences of concepts and skills that underlie mastery of school-based mathematics learning goals. As it was mentioned earlier, each of these learning sequences was designed to capture the spontaneous development of normative populations, and it is expected that children who have superior or impoverished learning opportunities might progress through these sequences at a somewhat faster or slower rate. Similarly, it is expected that mastery of any concept or skill task may be demonstrated at a somewhat earlier age if task demands are simplified and/or if performance is scaffolded or otherwise supported (e.g., with adult guidance).

The teaching methods and learning structures that are encouraged in the Number Worlds program (e.g., the use of small learning groups and the differentiation of instruction) are highly consistent with the theory, but they put heavy demands on teachers, even with the provision of multiple supports to enact this form of teaching. Informal research suggests that learning gains for students vary according to the degree to which teachers implement these recommendations, but, even with low levels of implementation, these gains are still higher than those achieved with other mathematics programs. I will close this article with a speculation. One of the greatest strengths of the Number Worlds program is the fact that children universally respond very positively to it and voice their new-found enthusiasm for doing math in multiple ways (e.g., wanting to give up recess to continue playing the program games). The inclusion of games in the program probably has something to do with this, but a greater factor, I believe, is the appropriateness of the program for children's developing capabilities, which minimizes the extent to which they will fail and feel stupid in math and which maximizes their willingness to engage in the process of doing math. This feature of the program is a product of the theory on which the program was based and, along with research that demonstrates learning gains for users of the program (see Griffin, 2005a, 2005b, 2007b; Griffin & Case, 1996, 1997 for a review of this research), it attests to the value of theory-based mathematics programs in schools.

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